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2006 J. Phys. A: Math. Gen. 39 7457

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The finite-temperature relativistic Landau problem and the relativistic quantum Hall effect*

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Received 16 November 2005

Published 23 May 2006

Online at stacks.iop.org/JPhysA/39/7457

Abstract

This paper presents a study of the free energy and particle density of the relativistic Landau problem, and their relevance to the quantum Hall effect. First we study the zero-temperature Casimir energy and fermion number for Dirac fields in a (2+1)-dimensional Minkowski spacetime, in the presence of a uniform magnetic field perpendicular to the spatial manifold. Then, we go to the finite-temperature problem, with a chemical potential, introduced as a uniform zero component of the gauge potential. By performing a Lorentz boost, we obtain Hall's conductivity in the case of crossed electric and magnetic fields.

PACS numbers: 11.10.Wx, 02.30.Sa

1. Introduction

The quantization of the Hall conductivity [1] is a remarkable quantum phenomenon, which occurs in two-dimensional electron systems, at low temperatures and strong perpendicular magnetic fields. Most proposed explanations for this phenomenon (for a review see, for instance, [2]) rely on one-particle theory and make use of the Kubo formula concerning the conductivity as a linear response function to the external field [3].

It is the aim of this paper to show that, in the context of relativistic field theory, the quantization of the Hall conductivity in multiples of the magnetic flux arises as a consequence of the spin-statistics theorem, which is a straightforward outcome of such a theory.

After this calculation was finished, a very interesting, recently published [4], calculation of the Hall conductivity in relativistic systems was brought to our knowledge. We will compare our results to those obtained in this recent publication in our final comments.

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In section 2 we present the theory of Dirac fields in 2+1 Minkowski spacetime, interacting with a magnetic background field perpendicular to the spatial plane, and evaluate the vacuum expectation values of the energy and fermion density.

Section 3 contains the generalities of the same theory in Euclidean three-dimensional space, and presents the eigenvalues of the corresponding Dirac operator. From such eigenvalues, the partition function is evaluated in section 4.

Section 5 contains the resulting free energy and mean particle density at finite temperature, and a study of their zero-temperature limit. In the same section, we perform an adequate Lorentz boost in order to consider the problem of fermions interacting with crossed electric and magnetic fields, and obtain the Hall conductivity.

Some final comments about the case in which the chemical potential μ coincides with an energy level are presented in section 6. In the same section, we discuss the role that experimental results on the quantum Hall effect can play, as an arena for testing the physical relevance of the phase of the determinant or, equivalently, of the multiplicative anomaly, in view of our discrepancies with [4].

A very sketchy presentation of the present study can be found in [5].

2. Zero-temperature problem

We will use the metric $(-, +, +)$, natural units $\hbar = c = 1$, and choose the following representation for the Dirac matrices: namely,

$$\gamma_M^0 = i\sigma_3, \quad \gamma_M^1 = \sigma_2 \quad \text{and} \quad \gamma_M^2 = \sigma_1. \quad (1)$$

The Hamiltonian can be determined from the solutions of the Dirac equation $(i\partial - e\mathcal{A})\Psi = 0$, where $-e$ is the negative charge of the electron. In the Landau gauge $A = (0, 0, Bx)$, with $B > 0$. Thus, after setting $\Psi(t, x, y) = e^{-iEt}\psi(x, y)$, we get the Hamiltonian $H = i\sigma_1\partial_x - i\sigma_2\partial_y + \sigma_2eBx$. The corresponding eigenvalues can be determined from

$$\begin{pmatrix} -E & i\partial_x - \partial_y - ieBx \\ i\partial_x + \partial_y + ieBx & -E \end{pmatrix} \psi(x, y) = 0. \quad (2)$$

In order to solve the above equation, we take

$$\psi_k(x, y) = \begin{pmatrix} \varphi_k(x, y) \\ \chi_k(x, y) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{iky}\varphi_k(x) \\ e^{iky}\chi_k(x) \end{pmatrix}. \quad (3)$$

This leads to the following system of first-order equations:

$$(i\partial_x - ik - ieBx)\chi_k = E\varphi_k \quad (i\partial_x + ik + ieBx)\varphi_k = E\chi_k. \quad (4)$$

We first solve for the zero mode $E_0 = 0$. In this case, both equations decouple and, after imposing that the eigenfunctions be well-behaved for $x \rightarrow \pm\infty$ we obtain, up to normalization,

$$\psi_k(x) = \begin{pmatrix} (eB/\pi)^{1/4} \exp\left\{-\frac{1}{2}eB(x+k/eB)^2\right\} \\ 0 \end{pmatrix}. \quad (5)$$

In the case $E \neq 0$, the normalized eigenfunctions which are well-behaved for all values of x can be written in terms of Hermite's polynomials $H_n(u)$ [6] as follows,

$$\psi_{k,n}^\pm(x) = \frac{(eB/\pi)^{1/4}}{\sqrt{n!2^{n+1}}} \exp\left\{-\frac{1}{2}eB(x+k/eB)^2\right\} \begin{pmatrix} \pm H_n(\sqrt{eB}(x+k/eB)) \\ i\sqrt{2n}H_{n-1}(\sqrt{eB}(x+k/eB)) \end{pmatrix}, \quad (6)$$

with corresponding eigenvalues

$$E_n = \pm\sqrt{2neB}, \quad n = 1, \dots, \infty. \quad (7)$$

We note that each eigenvalue exhibits the well-known Landau’s degeneracy per unit area: namely,

$$\Delta_L = \frac{eB}{2\pi}. \tag{8}$$

Had we chosen the other non-equivalent representation of the gamma matrices in 2 + 1 dimensions, exactly the same spectrum would have been obtained, the only difference being the chirality of eigenfunctions.

The vacuum expectation value of the energy per unit area, defined through a ζ -function regularization (see, for example, [7], and references therein), is given by

$$E_C = -\frac{\Delta_L}{2} \sum_{E_n \neq 0} |E_n|^{-s} \Big|_{s=-1}. \tag{9}$$

In the present case, we have (α is an arbitrary parameter with mass dimension, introduced to render the complex powers dimensionless)

$$E_C(B) = -\frac{\Delta_L \alpha}{2} 2 \sum_{n=1}^{\infty} \left(\frac{\sqrt{2neB}}{\alpha} \right)^{-s} \Big|_{s=-1} = -\Delta_L \sqrt{2eB} \zeta_R \left(-\frac{1}{2} \right). \tag{10}$$

Always in the ζ -function regularization framework, the fermion number is [8]

$$N(B) = -\frac{\Delta_L}{2} \left(\sum_{E_n > 0} |E_n|^{-s} - \sum_{E_n < 0} |E_n|^{-s} \right) \Big|_{s=0} + N_0,$$

where N_0 is the contribution coming from zero modes.

In our case, the nonvanishing spectrum is symmetric. So, only the zero mode, which is charge self-conjugate, contributes. This gives as a result [8]

$$N(B) = \pm \frac{\Delta_L}{2} \tag{11}$$

or, equivalently, for the vacuum expectation value of the charge density,

$$j^0(B) = \mp e \frac{\Delta_L}{2}. \tag{12}$$

The sign indetermination is a natural consequence of the twofold vacuum degeneracy.

3. The theory at finite temperature with chemical potential

In order to study the effect of temperature, we go to Euclidean space, with the metric (+, +, +). To this end, we take the Euclidean gamma matrices to be $\gamma_0 = i\gamma_M^0 = -\sigma_3$, $\gamma_1 = \gamma_M^1 = \sigma_2$, $\gamma_2 = \gamma_M^2 = \sigma_1$. We will follow [9] in introducing the chemical potential as an imaginary $A_0 = -i\frac{\mu}{e}$ in Euclidean space. Thus, the partition function in the grand-canonical ensemble is given by

$$\ln Z = \ln \det(i\partial - e\mathcal{A}). \tag{13}$$

In order to evaluate the partition function in the ζ -regularization approach [10], we first determine the eigenfunctions, and the corresponding eigenvalues, of the Dirac operator, in the same gauge used in the previous section, i.e., we solve

$$[-i\sigma_3(\partial_\tau + \mu) + i\sigma_2\partial_x + \sigma_1(i\partial_y - eBx)]\Psi = \omega\Psi, \tag{14}$$

or, after writing $\Psi(\tau, x, y) = \begin{pmatrix} \Phi(\tau, x, y) \\ \Xi(\tau, x, y) \end{pmatrix}$,

$$\begin{pmatrix} -i(\partial_\tau + \mu) & \partial_x + i\partial_y - eBx \\ -\partial_x + i\partial_y - eBx & i(\partial_\tau + \mu) \end{pmatrix} \begin{pmatrix} \Phi(\tau, x, y) \\ \Xi(\tau, x, y) \end{pmatrix} = \omega \begin{pmatrix} \Phi(\tau, x, y) \\ \Xi(\tau, x, y) \end{pmatrix}. \tag{15}$$

In order to satisfy antiperiodic boundary conditions in the τ direction, we propose

$$\Psi_{k,l}(\tau, x, y) = \frac{e^{i\lambda_l \tau} e^{iky}}{\sqrt{2\pi\beta}} \psi_{k,l}(x), \tag{16}$$

with

$$\lambda_l = (2l + 1) \frac{\pi}{\beta}, \tag{17}$$

where $\beta = \frac{1}{T}$ is the inverse temperature.

After doing so, and writing

$$\psi_{k,l}(x) = \begin{pmatrix} \varphi_{k,l}(x) \\ \chi_{k,l}(x) \end{pmatrix},$$

we have, for each k, l ,

$$(\partial_x - k - eBx)\chi_{k,l} = (\omega - \tilde{\lambda}_l)\varphi_{k,l} \quad (-\partial_x - k - eBx)\varphi_{k,l} = (\omega + \tilde{\lambda}_l)\chi_{k,l}, \tag{18}$$

where $\tilde{\lambda}_l = \lambda_l - i\mu$.

There are two types of eigenvalues and corresponding eigenfunctions:

- (1) $\omega_l = \tilde{\lambda}_l$, with $l = -\infty, \dots, \infty$, and corresponding normalized eigenfunctions

$$\psi_{k,l}(x) = \begin{pmatrix} \left(\frac{eB}{\pi}\right)^{\frac{1}{4}} e^{-\frac{eB}{2}(x+\frac{k}{eB})^2} \\ 0 \end{pmatrix}. \tag{19}$$

Note that these eigenvalues are not square roots of the eigenvalues of the squared operator. They will eventually lead to a ‘spectral asymmetry’¹ and, thus, to a phase of the determinant, which will be studied in detail in the next section.

- (2) $\omega_{l,n} = \pm\sqrt{\tilde{\lambda}_l^2 + 2neB}$, with $n = 1, \dots, \infty, l = -\infty, \dots, \infty$, and corresponding normalized eigenfunctions

$$\psi_{k,l,n}(x) = A_{k,l,n} \begin{pmatrix} -\frac{(\omega_{l,n} + \tilde{\lambda}_l)}{2n\sqrt{eB}} e^{-\frac{eB}{2}(x+\frac{k}{eB})^2} H_n(\sqrt{eB}(x + \frac{k}{eB})) \\ e^{-\frac{eB}{2}(x+\frac{k}{eB})^2} H_{n-1}(\sqrt{eB}(x + \frac{k}{eB})) \end{pmatrix}, \tag{20}$$

where

$$A_{k,l,n} = \left(\frac{eB}{\pi}\right)^{\frac{1}{4}} \frac{2^{\frac{1-n}{2}}}{[(n-1)!]^{\frac{1}{2}}} \left[\frac{2neB}{2neB + |\omega_{l,n} + \tilde{\lambda}_l|^2} \right]^{\frac{1}{2}}.$$

In all cases, the degeneracy per unit area is again given by Δ_L in equation (8). Choosing the other non-equivalent representation of the gamma matrices leads to a change in the eigenvalues of type 1, which are replaced by $\omega_l = -\tilde{\lambda}_l$. However, as will be discussed below, this does not lead to a change in our physical predictions.

¹ Here, quotation marks are due to the fact that, the Dirac operator not being self-adjoint, we have a complex spectrum.

4. Evaluation of the partition function at finite temperature and chemical potential

The partition function, in the ζ -regularization scheme [10], is given by

$$\log \mathcal{Z} = -\frac{d}{ds} \Big|_{s=0} \zeta \left(s, \frac{i\partial - e\mathcal{A}}{\alpha} \right). \tag{21}$$

As in the previous section, α is a parameter with mass dimension, introduced to render the ζ -function dimensionless.

We must consider two contributions to $\log \mathcal{Z}$, respectively coming from eigenvalues of type 1 and 2 in the previous section, i.e.,

$$\Delta_1(\mu) = -\frac{d}{ds} \Big|_{s=0} \zeta_1(s, \mu), \tag{22}$$

where

$$\zeta_1(s, \mu) = \Delta_L \sum_{l=-\infty}^{\infty} \left[(2l+1) \frac{\pi}{\alpha\beta} - i \frac{\mu}{\alpha} \right]^{-s}, \tag{23}$$

and

$$\Delta_2(\mu, B) = -\frac{d}{ds} \Big|_{s=0} \zeta_2(s, \mu, B), \tag{24}$$

where

$$\zeta_2(s, \mu, B) = (1 + (-1)^{-s}) \Delta_L \sum_{\substack{n=1 \\ l=-\infty}}^{\infty} \left[\frac{2neB}{\alpha^2} + \left((2l+1) \frac{\pi}{\alpha\beta} - i \frac{\mu}{\alpha} \right)^2 \right]^{-\frac{s}{2}}. \tag{25}$$

The contribution $\Delta_1(\mu)$ can be evaluated at once for the whole μ -range. The analytic extension of $\zeta_1(s, \mu)$ can be achieved as follows (for a similar calculation, see [11]):

$$\begin{aligned} \zeta_1(s, \mu) &= \Delta_L \sum_{l=-\infty}^{\infty} \left[(2l+1) \frac{\pi}{\alpha\beta} - i \frac{\mu}{\alpha} \right]^{-s} \\ &= \Delta_L \left(\frac{2\pi}{\alpha\beta} \right)^{-s} \left[\sum_{l=0}^{\infty} \left[\left(l + \frac{1}{2} \right) - i \frac{\mu\beta}{2\pi} \right]^{-s} + \sum_{l=0}^{\infty} \left[- \left(l + \frac{1}{2} \right) - i \frac{\mu\beta}{2\pi} \right]^{-s} \right] \\ &= \Delta_L \left(\frac{2\pi}{\alpha\beta} \right)^{-s} \left[\zeta_H \left(s, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + \sum_{l=0}^{\infty} \left[- \left(l + \frac{1}{2} \right) - i \frac{\mu\beta}{2\pi} \right]^{-s} \right]. \end{aligned} \tag{26}$$

Now, in order to write the second term as a Hurwitz ζ , we must relate the eigenvalues with negative real part to those with positive one without, in so doing, going through zeros in the argument of the power. Otherwise stated, we must select a cut in the complex ω plane [12]. This requirement determines a definite value of $(-1)^{-s}$, i.e., $(-1)^{-s} = \exp(i\pi \operatorname{sign}(\mu)s)$. Taking this into account, we finally have

$$\zeta_1(s, \mu) = \Delta_L \left(\frac{2\pi}{\beta\alpha} \right)^{-s} \left[\zeta_H \left(s, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + e^{i\pi \operatorname{sign}(\mu)s} \zeta_H \left(s, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) \right]. \tag{27}$$

From this last expression, the contribution $\Delta_1(\mu)$ to $\log \mathcal{Z}$ can be obtained. It is given by

$$\begin{aligned} \Delta_1(\mu) &= -\Delta_L \left[\zeta'_H \left(0, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + \zeta'_H \left(0, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) + i\pi \operatorname{sign}(\mu) \zeta_H \left(0, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) \right] \\ &= \Delta_L \left\{ \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right) - \frac{|\mu|\beta}{2} \right\}. \end{aligned} \tag{28}$$

As commented in advance, the other non-equivalent representation of the gamma matrices leads to the same result for this contribution. In fact, even though this part of the spectrum changes sign, such change is compensated by the selection of the cut in the ω plane. In this case, one has

$$\zeta_1(s, \mu) = \Delta_L \left(\frac{2\pi}{\beta\alpha}\right)^{-s} \left[\zeta_H\left(s, \frac{1}{2} + \frac{i\mu\beta}{2\pi}\right) + e^{-i\pi\text{sign}(\mu)s} \zeta_H\left(s, \frac{1}{2} - \frac{i\mu\beta}{2\pi}\right) \right],$$

which also leads to (28).

The analytic extension of $\zeta_2(s, \mu, B)$ requires a separate consideration of different μ -ranges. We study in detail three of these cases. The generalization to arbitrary μ -ranges will be evident from these results.

4.1. $\mu^2 < 2eB$

$$\zeta_2(s, \mu, B) = (1 + (-1)^{-s})\Delta_L \sum_{l=-\infty}^{\infty} \left[\frac{2neB}{\alpha^2} + \left((2l+1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha} \right)^2 \right]^{-\frac{s}{2}}. \tag{29}$$

Making use of the Mellin transform, this can be written as

$$\begin{aligned} \zeta_2(s, \mu, B) &= \frac{(1 + (-1)^{-s})\Delta_L}{\Gamma(\frac{s}{2})} \int_0^\infty dt t^{\frac{s}{2}-1} \\ &\times \sum_{l=-\infty}^{\infty} \exp\left(-t \left[\frac{2neB}{\alpha^2} + \left((2l+1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha} \right)^2 \right]\right) \end{aligned} \tag{30}$$

or, equivalently,

$$\begin{aligned} \zeta_2(s, \mu, B) &= \frac{(1 + (-1)^{-s})\Delta_L}{\Gamma(\frac{s}{2})} \sum_{n=1}^{\infty} \int_0^\infty dt t^{\frac{s}{2}-1} \exp\left(-t \left[\frac{2neB}{\alpha^2} + \left(\frac{\pi}{\alpha\beta} - \frac{i\mu}{\alpha} \right)^2 \right]\right) \\ &\times \Theta_3\left(\frac{-2t}{\alpha\beta} \left(\frac{\pi}{\alpha\beta} - \frac{i\mu}{\alpha} \right), \frac{4\pi t}{(\alpha\beta)^2}\right), \end{aligned} \tag{31}$$

where we have used the definition of the Jacobi θ -function

$$\Theta_3(z, x) = \sum_{l=-\infty}^{\infty} e^{-\pi x l^2} e^{2\pi i z l}. \tag{32}$$

To proceed, we will use the inversion formula for the Jacobi function

$$\Theta_3(z, x) = \frac{1}{\sqrt{x}} e^{\frac{\pi z^2}{x}} \Theta_3\left(\frac{z}{ix}, \frac{1}{x}\right), \tag{33}$$

thus getting

$$\zeta_2(s, \mu, B) = \frac{(1 + (-1)^{-s})\Delta_L\alpha\beta}{2\sqrt{\pi}\Gamma(\frac{s}{2})} \sum_{n=1}^{\infty} \int_0^\infty dt t^{\frac{s-1}{2}-1} e^{-t\frac{2neB}{\alpha^2}} \Theta_3\left(\frac{i}{2} + \frac{\mu\beta}{2\pi}, \frac{(\alpha\beta)^2}{4\pi t}\right). \tag{34}$$

Applying once more the definition (32), we have

$$\begin{aligned} \zeta_2(s, \mu, B) &= \frac{(1 + (-1)^{-s})\Delta_L\alpha\beta}{2\sqrt{\pi}\Gamma(\frac{s}{2})} \left\{ \int_0^\infty dt t^{\frac{s-1}{2}-1} \sum_{n=1}^{\infty} e^{-t\frac{2neB}{\alpha^2}} \right. \\ &\left. + 2 \int_0^\infty dt t^{\frac{s-1}{2}-1} \sum_{n,l=1}^{\infty} (-1)^l \cosh(\mu\beta l) e^{-t\frac{2neB}{\alpha^2} - \frac{(\alpha\beta l)^2}{4t}} \right\}. \end{aligned} \tag{35}$$

After performing the integrals [6], we obtain

$$\zeta_2(s, \mu, B) = \frac{(1 + (-1)^{-s})\Delta_L\alpha\beta}{2\sqrt{\pi}\Gamma(\frac{s}{2})} \left[\Gamma\left(\frac{s-1}{2}\right) \left(\frac{2eB}{\alpha^2}\right)^{\frac{1-s}{2}} \zeta_R\left(\frac{s-1}{2}\right) + 4 \sum_{n,l=1}^{\infty} (-1)^l \left(\frac{l^2\alpha^4\beta^2}{8neB}\right)^{\frac{s-1}{4}} \cosh(\mu\beta l) K_{\frac{s-1}{2}}(\sqrt{2neB\beta^2 l^2}) \right] \tag{36}$$

or, making the simple pole of the Γ -function explicit

$$\zeta_2(s, \mu, B) = \frac{(1 + (-1)^{-s})\Delta_L\alpha\beta s}{4\sqrt{\pi}\Gamma(\frac{s+2}{2})} \left[\Gamma\left(\frac{s-1}{2}\right) \left(\frac{2eB}{\alpha^2}\right)^{\frac{1-s}{2}} \zeta_R\left(\frac{s-1}{2}\right) + 4 \sum_{n,l=1}^{\infty} (-1)^l \left(\frac{l^2\alpha^4\beta^2}{8neB}\right)^{\frac{s-1}{4}} \cosh(\mu\beta l) K_{\frac{s-1}{2}}(\sqrt{2neB\beta^2 l^2}) \right]. \tag{37}$$

From this expression, the contribution Δ_2 to the partition function can be readily obtained, since the factor accompanying s is finite at $s = 0$

$$\Delta_2(\mu, B) = \frac{-\Delta_L\beta}{2\sqrt{\pi}} \left[\Gamma\left(-\frac{1}{2}\right) \sqrt{2eB} \zeta_R\left(-\frac{1}{2}\right) + 4 \sum_{n,l=1}^{\infty} (-1)^l \left(\frac{l^2\beta^2}{8neB}\right)^{-\frac{1}{4}} \cosh(\mu\beta l) K_{-\frac{1}{2}}(\sqrt{2neB\beta^2 l^2}) \right]. \tag{38}$$

After using that

$$K_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x},$$

it can be written as

$$\Delta_2(\mu, B) = \Delta_L\beta \left[\sqrt{2eB} \zeta_R\left(-\frac{1}{2}\right) - \frac{2}{\beta} \sum_{n,l=1}^{\infty} \frac{(-1)^l}{l} \cosh(\mu\beta l) e^{-\sqrt{2neB}\beta l} \right]. \tag{39}$$

The sum over l can be explicitly performed, to obtain

$$\Delta_2(\mu, B) = \Delta_L\beta \left[\sqrt{2eB} \zeta_R\left(-\frac{1}{2}\right) + \frac{1}{\beta} \sum_{n=1}^{\infty} \log(1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta}) \right]. \tag{40}$$

Finally, adding the contributions given by equations (28) and (40) we obtain, for the partition function in the range $\mu^2 \leq 2eB$,

$$\log Z = \Delta_L \left\{ \log\left(2 \cosh\left(\frac{\mu\beta}{2}\right)\right) - \frac{|\mu|\beta}{2} + \beta\sqrt{2eB} \zeta_R\left(-\frac{1}{2}\right) + \sum_{n=1}^{\infty} \log(1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta}) \right\}. \tag{41}$$

4.2. $2eB < \mu^2 < 4eB$

As before, we have

$$\zeta_2(s, \mu, B) = (1 + (-1)^{-s}) \Delta_L \sum_{\substack{n=1 \\ l=-\infty}}^{\infty} \left[\frac{2neB}{\alpha^2} + \left((2l+1) \frac{\pi}{\alpha\beta} - i \frac{\mu}{\alpha} \right)^2 \right]^{-\frac{s}{2}}. \quad (42)$$

However, in this range of μ , the contribution due to $n = 1$ is given by

$$\Delta_2^{n=1}(\mu, B) = - \frac{d}{ds} \Big|_{s=0} \zeta_2^{n=1}(s, \mu, B), \quad (43)$$

where

$$\zeta_2^{n=1}(s, \mu, B) = (1 + (-1)^{-s}) \Delta_L \sum_{l=-\infty}^{\infty} \left[\frac{2eB}{\alpha^2} + \left((2l+1) \frac{\pi}{\alpha\beta} - i \frac{\mu}{\alpha} \right)^2 \right]^{-\frac{s}{2}}. \quad (44)$$

The analytic extension of this expression must be performed in a different way. In fact, the expression cannot be written in terms of a unique Mellin transform, since its real part is not always positive (note, in connection with this, that for $n = 1$ equation (37) diverges). Instead, it can be evaluated as follows:

$$\begin{aligned} \zeta_2^{n=1}(s, \mu, B) &= \frac{(1 + (-1)^{-s})}{\alpha^{-s}} \Delta_L \sum_{l=0}^{\infty} \left[2eB + \left((2l+1) \frac{\pi}{\beta} - i\mu \right)^2 \right]^{-\frac{s}{2}} + \mu \rightarrow -\mu \\ &= \frac{(1 + (-1)^{-s})}{\alpha^{-s}} \Delta_L \sum_{l=0}^{\infty} \left\{ \left[i\sqrt{2eB} + (2l+1) \frac{\pi}{\beta} - i\mu \right]^{-\frac{s}{2}} \right. \\ &\quad \left. \times \left[-i\sqrt{2eB} + (2l+1) \frac{\pi}{\beta} - i\mu \right]^{-\frac{s}{2}} \right\} + \mu \rightarrow -\mu. \end{aligned} \quad (45)$$

This can be written as a product of two Mellin transforms

$$\begin{aligned} \zeta_2^{n=1}(s, \mu, B) &= \frac{(1 + (-1)^{-s})}{\alpha^{-s} [\Gamma(\frac{s}{2})]^2} \Delta_L \sum_{l=0}^{\infty} \int_0^{\infty} dt t^{\frac{s}{2}-1} \exp \left(- \left[(2l+1) \frac{\pi}{\beta} - i\mu + i\sqrt{2eB} \right] t \right) \\ &\quad \times \int_0^{\infty} dz z^{\frac{s}{2}-1} \exp \left(- \left[(2l+1) \frac{\pi}{\beta} - i\mu - i\sqrt{2eB} \right] z \right) + \mu \rightarrow -\mu \end{aligned} \quad (46)$$

or, after a change of variables

$$\begin{aligned} \zeta_2^{n=1}(s, \mu, B) &= \frac{(1 + (-1)^{-s})}{2\alpha^{-s} [\Gamma(\frac{s}{2})]^2} \Delta_L \sum_{l=0}^{\infty} \int_0^{\infty} dz \int_{-z}^z dt \left[\frac{z^2 - t^2}{4} \right]^{\frac{s}{2}-1} \\ &\quad \times e^{-[(2l+1)\frac{\pi}{\beta} - i\mu]z} e^{-i\sqrt{2eB}t} + \mu \rightarrow -\mu. \end{aligned} \quad (47)$$

Now, calling $x = \frac{t}{z}$, one has

$$\begin{aligned} \zeta_2^{n=1}(s, \mu, B) &= \frac{(1 + (-1)^{-s})}{2\alpha^{-s} [\Gamma(\frac{s}{2})]^2} \Delta_L \sum_{l=0}^{\infty} \int_0^{\infty} dz z \left[\frac{z^2}{4} \right]^{\frac{s}{2}-1} e^{-[(2l+1)\frac{\pi}{\beta} - i\mu]z} \\ &\quad \times \int_{-1}^1 dx [1 - x^2]^{\frac{s}{2}-1} e^{-i\sqrt{2eB}zx} + \mu \rightarrow -\mu. \end{aligned} \quad (48)$$

Finally, the x -integral and the sum of the geometric series can be performed to obtain

$$\zeta_2^{n=1}(s, \mu, B) = \frac{(1 + (-1)^{-s})\sqrt{\pi}}{2\alpha^{-s}\Gamma(\frac{s}{2})} \Delta_L (2\sqrt{2eB})^{\frac{1-s}{2}} \times \int_0^\infty dz z^{\frac{s-1}{2}} J_{\frac{s-1}{2}}(\sqrt{2eB}z) \frac{e^{i\mu z}}{\sinh(\frac{\pi z}{\beta})} + \mu \rightarrow -\mu. \tag{49}$$

Now, the integral in this expression diverges at $z = 0$. In order to isolate such divergence, we add and subtract the first term in the series expansion of the Bessel function, thus getting the following two pieces,

$$\zeta_{2,(1)}^{n=1}(s, \mu, B) = \frac{(1 + (-1)^{-s})\sqrt{\pi}s}{4\alpha^{-s}\Gamma(\frac{s}{2} + 1)} \Delta_L (2\sqrt{2eB})^{\frac{1-s}{2}} \times \int_0^\infty dz z^{\frac{s-1}{2}} \left[J_{\frac{s-1}{2}}(\sqrt{2eB}z) - \frac{(\frac{\sqrt{2eB}z}{2})^{\frac{s-1}{2}}}{\Gamma(\frac{s+1}{2})} \right] \frac{e^{i\mu z}}{\sinh(\frac{\pi z}{\beta})} + \mu \rightarrow -\mu, \tag{50}$$

and

$$\zeta_{2,(2)}^{n=1}(s, \mu, B) = \frac{(1 + (-1)^{-s})\sqrt{\pi}}{2^s\alpha^{-s}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})} \Delta_L \int_0^\infty dz z^{s-1} \frac{e^{i\mu z}}{\sinh(\frac{\pi z}{\beta})} + \mu \rightarrow -\mu. \tag{51}$$

The contribution of equation (50) to the partition function, defined as in equation (43), can be easily evaluated by noting that the factor multiplying s is finite at $s = 0$. Thus, one has

$$\Delta_{2,(1)}^{n=1}(\mu, B) = -\Delta_L \int_0^\infty dz z^{-1} [\cos(\sqrt{2eB}z) - 1] \frac{e^{i\mu z}}{\sinh(\frac{\pi z}{\beta})} + \mu \rightarrow -\mu, \tag{52}$$

where we have used that $J_{-\frac{1}{2}}(\sqrt{2eB}z) = \sqrt{\frac{2}{\pi\sqrt{2eB}z}} \cos(\sqrt{2eB}z)$. Now, in the term with $\mu \rightarrow -\mu$, one can change $z \rightarrow -z$ to obtain

$$\Delta_{2,(1)}^{n=1}(\mu, B) = -\Delta_L \int_{-\infty}^\infty dz z^{-1} [\cos(\sqrt{2eB}z) - 1] \frac{e^{i\mu z}}{\sinh(\frac{\pi z}{\beta})}. \tag{53}$$

This last integral is easy to evaluate in the complex plane, by carefully taking into account the sign of μ , as well as the fact that $2eB < \mu^2$ in closing the integration path, to obtain

$$\Delta_{2,(1)}^{n=1}(\mu, B) = -2\Delta_L \sum_{l=1}^\infty \left[\frac{(-1)^l}{l} \cosh(\sqrt{2eB}\beta l) e^{-|\mu|\beta l} + \frac{(-1)^{l+1}}{l} e^{-|\mu|\beta l} \right] \tag{54}$$

or, after summing the series,

$$\Delta_{2,(1)}^{n=1}(\mu, B) = \Delta_L \left\{ \log(1 + e^{-2|\mu|\beta} + 2 \cosh(\sqrt{2eB}\beta) e^{-|\mu|\beta}) + |\mu|\beta - 2 \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right) \right\}. \tag{55}$$

In order to obtain the contribution coming from (51), the integral can be evaluated for $\text{Re } s > 1$, which gives

$$\zeta_{2,(2)}^{n=1}(s, \mu, B) = \frac{(1 + (-1)^{-s})\Gamma(s)\sqrt{\pi}(\alpha\beta)^s \Delta_L}{(2\pi)^s 2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})} \times \left[\zeta_H \left(s, \frac{1}{2} \left(1 - \frac{i\mu\beta}{\pi} \right) \right) + \zeta_H \left(s, \frac{1}{2} \left(1 + \frac{i\mu\beta}{\pi} \right) \right) \right], \tag{56}$$

where $\zeta_H(s, x)$ is the Hurwitz ζ -function. The contribution to the partition function can now be evaluated by using that $\zeta_H(0, \frac{1}{2}(1 - \frac{i\mu\beta}{\pi})) + \zeta_H(0, \frac{1}{2}(1 + \frac{i\mu\beta}{\pi})) = 0$ and the well-known value of $-\frac{d}{ds} \Big|_{s=0} \zeta_H(s, x)$ [6], to obtain

$$\Delta_{2,(2)}^{n=1}(\mu, B) = 2\Delta_L \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right). \quad (57)$$

Summing up the contributions in equations (28), (55) and (57), as well as the contribution coming from $n \geq 2$, evaluated as in the previous subsection, one gets for the partition function

$$\begin{aligned} \log Z = \Delta_L \left\{ \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right) + \frac{|\mu|\beta}{2} + \log(1 + e^{-2|\mu|\beta} + 2 \cosh(\sqrt{2eB}\beta) e^{-|\mu|\beta}) \right. \\ \left. + \beta\sqrt{2eB} \left(\zeta_R \left(-\frac{1}{2} \right) - 1 \right) + \sum_{n=2}^{\infty} \log(1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta}) \right\}. \end{aligned} \quad (58)$$

At first sight, this result looks different from the one corresponding to $\mu^2 < 2eB$ (equation (41)). However, it is easy to see that both expressions coincide. The only difference is that (58) explicitly isolates the zero-temperature behaviour from finite-temperature corrections for this range of μ .

4.3. $\mu^2 = 2eB$

In this case, the analytical extension can be performed, exactly as in the previous subsection, up to equation (52). Then, the term with $\mu \rightarrow -\mu$ can be explicitly summed, which gives

$$\Delta_{2,(1)}^{n=1}(|\mu| = \sqrt{2eB}, B) = -\Delta_L \int_0^{\infty} dz z^{-1} [\cos(\sqrt{2eB}z) - 1] \frac{2 \cos(\sqrt{2eB}z)}{\sinh(\frac{\pi z}{\beta})}. \quad (59)$$

This integral can be found in [6], and it gives as a result

$$\Delta_{2,(1)}^{n=1}(|\mu| = \sqrt{2eB}, B) = \Delta_L \left[\log(\cosh(\sqrt{2eB}\beta)) - 2 \log \left(\cosh \left(\frac{\sqrt{2eB}\beta}{2} \right) \right) \right]. \quad (60)$$

The contribution $\Delta_{2,(2)}^{n=1}(|\mu| = \sqrt{2eB}, B)$ can be evaluated exactly as in the previous subsection. It is given by

$$\Delta_{2,(2)}^{n=1}(|\mu| = \sqrt{2eB}, B) = 2\Delta_L \log \left(2 \cosh \left(\frac{\sqrt{2eB}\beta}{2} \right) \right). \quad (61)$$

These two contributions, together with those coming from $n \geq 2$ and from equation (28), finally give for the partition function at this particular value of μ the same result as (41) or (58) evaluated at $\mu^2 = 2eB$.

Thus, for any range of μ , one has

$$\begin{aligned} \log Z = \Delta_L \left\{ \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right) - \frac{|\mu|\beta}{2} + \beta\sqrt{2eB} \zeta_R \left(-\frac{1}{2} \right) \right. \\ \left. + \sum_{n=1}^{\infty} \log(1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta}) \right\}, \end{aligned} \quad (62)$$

which is continuous, even when μ coincides with an energy level.

When this calculation was finished, it was pointed to us that, in [13], the partition function for Dirac fields interacting not only with the background magnetic field, but also among themselves through a Nambu–Jona–Lasinio term, had been obtained through a different regularization (see also [14]; for a calculation of parity-breaking corrections, see [15]). However, in the absence of this last interaction, the result in such reference differs from the present one for two reasons. In the first place, the authors of [13] were considering a reducible representation of the gamma matrices, which lead them to an overall factor of 2 with respect to (62). But, more important, in the same reference the determinant of the Dirac operator was evaluated as the square root of minus the squared Dirac operator, thus not including the factor coming from the phase of the determinant or, equivalently, from the multiplicative anomaly [16], which we did consider (see equation (28)). As we will discuss in detail in section 6, this leads to completely different predictions regarding the Hall conductivity.

5. Free energy and particle density

From equation (62), the free energy per unit area ($F = -\frac{1}{\beta} \log Z$) can be obtained². It is given by

$$F(\mu, B, \beta) = -\Delta_L \left\{ \frac{1}{\beta} \log \left(2 \cosh \left(\frac{\mu\beta}{2} \right) \right) - \frac{|\mu|}{2} + \sqrt{2eB} \zeta_R \left(-\frac{1}{2} \right) + \frac{1}{\beta} \sum_{n=1}^{\infty} \log \left(1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta} \right) \right\}. \tag{63}$$

Moreover, the free energy is continuous at $\mu^2 = 2keB, k = 0, \dots, \infty$. In the low-temperature limit one has

$$F(2keB < \mu^2 < 2(k+1)eB) \rightarrow_{\beta \rightarrow \infty} -\Delta_L \left\{ \sqrt{2eB} \left(\zeta_R \left(-\frac{1}{2} \right) - \sum_{n=1}^k \sqrt{n} \right) + k|\mu| \right\}.$$

Note that, for $k = 0$, this result coincides with the Casimir energy obtained in section 2, even for $\mu \neq 0$, but in this range, i.e., for μ less than the first Landau level, if positive, or greater than minus the first Landau level, if negative.

The mean particle density can be obtained, also from (62), as $N = \frac{1}{\beta} \frac{d}{d\mu} \log Z$. For nonzero temperature and arbitrary μ one has

$$N(\mu, B, \beta) = \Delta_L \left\{ \frac{1}{2} \left[\tanh \left(\frac{\mu\beta}{2} \right) - \text{sign}(\mu) \right] + \sum_{n=1}^{\infty} \frac{2 \sinh(\mu\beta) e^{-\sqrt{2neB}\beta}}{1 + e^{-2\sqrt{2neB}\beta} + 2 \cosh(\mu\beta) e^{-\sqrt{2neB}\beta}} \right\}. \tag{64}$$

For nonvanishing μ , the low-temperature limit differs, depending on the μ -range considered

$$N(2ekB < \mu^2 < 2e(k+1)B) \rightarrow_{\beta \rightarrow \infty} k\Delta_L \text{sign}(\mu),$$

where, as before, $k = \left[\frac{\mu^2}{2eB} \right]$.

This result was to be expected for particles with the statistic of fermions, since relativistic field theory naturally leads to the spin-statistics theorem. At zero temperature, μ is nothing

² Consistently with the comments in previous sections, all the results in this section and in the rest of this paper are independent from the representation of the gamma matrices chosen.

but the Fermi energy; for example, for $\mu > 0$, as μ grows past a Landau level, such level becomes entirely filled.

From the previous result, the mean value of the particle density at zero temperature can be obtained. After recovering units, one has

$$j^0(2ec^2\hbar Bk < \mu^2 < 2eBc^2\hbar(k+1)) = \frac{-kce^2B}{h} \text{sign}(\mu),$$

the other two components of the current density tri-vector being equal to zero in the absence of an electric field. Now, the zero-temperature limit of the same tri-vector in the presence of crossed homogeneous electric (F') and magnetic (B') fields can be retrieved, for $F' < cB'$, by performing a Lorentz boost with an absolute value of the velocity $\frac{F'}{B'}$. Suppose, for definiteness, that the homogeneous electric field points along the positive y -axis. Then, the velocity of the Lorentz boost must point along the negative x -axis, and the transformation gives as a result

$$j'^0 = \frac{-kce^2B'}{h} \text{sign}(\mu), \quad j'^x = \frac{-ke^2F'}{h} \text{sign}(\mu), \quad j'^y = 0.$$

As a consequence, the quantized zero-temperature Hall conductivity is

$$\sigma_{xy} = \frac{-ke^2}{h} \text{sign}(\mu).$$

6. Final comments

A comment is in order concerning the zero temperature value of the particle density in equation (64) when the chemical potential coincides with an energy level, i.e., for $\mu^2 = 2ekB, k = 0, \dots, \infty$. In all these cases, the operation of setting μ to its value does not commute with that of taking the $\beta \rightarrow \infty$ limit.

For instance, in the case $\mu = 0, N(0, B, \beta)$ is undefined at all temperatures, while $\lim_{\mu \rightarrow 0} [\lim_{\beta \rightarrow \infty} N(\mu^2 < 2eB, B, \beta)] = 0$.

On the other hand, each time $\mu^2 = 2ekB, k = 1, \dots, \infty$, one has $\lim_{\beta \rightarrow \infty} N(\pm\sqrt{2ekB}, B, \beta) = \pm\Delta_L(k - \frac{1}{2})$, while $\lim_{\mu \rightarrow \pm\sqrt{2ekB}} [\lim_{\beta \rightarrow \infty} N(\mu, B, \beta)]$ is undetermined.

As already commented in the introduction, after this calculation was finished, [4] was brought to our attention. Their prediction concerning the Hall conductivity differs from ours. In fact, the Hall conductivity in that paper contains an overall factor of 4 with respect to ours, which is due to the use of a reducible representation and an extra sum over two values of spin. This is a more or less trivial difference. But, more important, after this factor is removed, their Hall conductivity is quantized in half-integer units of magnetic flux. This originates the word 'unconventional' in the title of [4]. Our calculation, instead, leads to a Hall conductivity quantized in terms of integer quanta of flux density, which is entirely conventional, but obtained here from very first principles. The difference between both results is due to the inclusion, in our calculation, of the phase of the determinant (or, equivalently in this case, of the multiplicative anomaly). The interesting comment at this point is that experimental results on the integer quantum Hall effect can clarify the physical relevance of the multiplicative anomaly (some recent results [17, 18] seemingly favour 'unconventional' quantization).

Finally, we mention that the more realistic case of massive fermions is at present under study [19].

Acknowledgments

We thank Professors Paola Giacconi, Gerardo Rossini and Roberto Soldati for useful discussions. We also thank Professor V P Gusynin for bringing [4] and [13] to our knowledge. This work was partially supported by Universidad Nacional de La Plata, under grant no 11/X381.

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